Another way to prove a new identity is to show that it follows from other identities that we know already.

**Example 23** Prove the identity $\tan\alpha \cot\alpha = 1$.

*Solution.* From our table, we see that $\tan\alpha = \sin\alpha / \cos\alpha$. We also see that $\cot\alpha = \cos\alpha / \sin\alpha$. Therefore,

$$\tan\alpha \cot\alpha = \left(\frac{\sin\alpha}{\cos\alpha}\right)\left(\frac{\cos\alpha}{\sin\alpha}\right) = 1\,. \qquad \square$$

**Example 24** Show that $\tan^2\alpha + 1 = 1/\cos^2\alpha$.

*Solution.* We know that $\sin^2\alpha + \cos^2\alpha = 1$, so

$$\frac{\sin^2\alpha}{\cos^2\alpha} + \frac{\cos^2\alpha}{\cos^2\alpha} = \frac{1}{\cos^2\alpha},$$

or

$$\tan^2\alpha + 1 = \frac{1}{\cos^2\alpha}. \qquad \square$$

You will have a chance to practice both these techniques in the exercises below.

**Exercises**

1. Verify that $\sin^2\alpha + \cos^2\alpha = 1$ if $\alpha$ equals 30°, 45°, and 60°.

2. The sine of an angle is $\sqrt{5}/4$. Express in radical form the cosine of this angle.

3. The cosine of an angle is $2/3$. Express in radical form the sine of the angle.

4. The tangent of an angle is $1/\sqrt{3}$. Find the numerical value of the sine and cosine of this angle.

5. Prove the following identities for an acute angle $\alpha$:

   a) $\cot x \sin x = \cos x$.

b) $\dfrac{\tan x}{\sin x} = \dfrac{1}{\cos x}$.

c) $\cos^2\alpha - \sin^2\alpha = 2\cos^2\alpha - 1$.

d) $\dfrac{\sin\alpha}{1+\cos\alpha} = \dfrac{1-\cos\alpha}{\sin\alpha}$.

e) $(\sin^2\alpha + 2\cos^2\alpha - 1)/\cot^2\alpha = \sin^2\alpha$.

f) $\cos^2\alpha = 1/(1+\tan^2\alpha)$.

g) $\sin^2\alpha = 1/(\cot^2\alpha + 1)$.

h) $\dfrac{1-\cos\alpha}{1+\cos\alpha} = \left(\dfrac{\sin\alpha}{1+\cos\alpha}\right)^2$.

i) $\dfrac{\sin^3\alpha - \cos^3\alpha}{\sin\alpha - \cos\alpha} = 1 + \sin\alpha\cos\alpha$.

6. a) For which angles $\alpha$ is $\sin^4\alpha - \cos^4\alpha > \sin^2\alpha - \cos^2\alpha$?

   b) For which angles $\alpha$ is $\sin^4\alpha - \cos^4\alpha \geq \sin^2\alpha - \cos^2\alpha$?

7. If $\tan\alpha = 2/5$, find the numerical value of $2\sin\alpha\cos\alpha$.

8. a) If $\tan\alpha = 2/5$, find the numerical value of $\cos^2\alpha - \sin^2\alpha$.

   b) If $\tan\alpha = r$, write an expression in terms of $r$ that represents the value of $\cos^2\alpha - \sin^2\alpha$.

9. If $\tan\alpha = 2/5$, find the numerical value of $\dfrac{\sin\alpha - 2\cos\alpha}{\cos\alpha - 3\sin\alpha}$.

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10. If $\tan\alpha = 2/5$, and $a$, $b$, $c$, $d$ are arbitrary rational numbers, with $5c + 2d \neq 0$, show that $\dfrac{a\sin\alpha + b\cos\alpha}{c\cos\alpha + d\sin\alpha}$ is a rational number.

11. For what value of $\alpha$ is the value of the expression $(\sin\alpha + \cos\alpha)^2 + (\sin\alpha - \cos\alpha)^2$ as large as possible?

## 5 Identities with secant and cosecant

While we do not often have to use the secant and cosecant, it is often convenient to express the fundamental identities above in terms of these two ratios. We can always restate the results as desired, using the fact that $\sec\alpha = 1/\cos\alpha$ and $\csc\alpha = 1/\sin\alpha$.